AMS Foundations Exam - Part A, January 2019

Name: ________________________________ ID Num. ____________________

Part A: ______ / 75

Part B: ______ / 75 Total: ______ / 150

This component of the exam (Part A) consists of two sections (Linear Algebra and Advanced Calculus) with four problems in each. Each question is worth 25 points; choose **THREE** questions to answer from **EACH** section. Each problem should be solvable in approximately 20 minutes or less. Provide your answer in the space provided, and show all work. If extra sheets are used, place them inside the booklet and note on the cover page how many additional pages are included.

Good Luck!

Section 1: Linear Algebra

Choose three of the four problems to solve.

1. Prove each of the following statements, where R is the set of real, symmetric matrices: $R = \left\{ \mathbf{Q} \in \mathbb{R}^{n \times n} : \mathbf{Q}^T = \mathbf{Q}, \forall n \in \mathbb{N} \right\}$.

   (a) For all $\mathbf{Q} \in R$, all eigenvalues are real.

   (b) For all $\mathbf{Q} \in R$, the eigenvectors of distinct eigenvalues are orthogonal.

2. Find the general form of the inverse of $\mathbf{C}$:

$$\mathbf{C} = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}, \quad \forall a, b, c \in \mathbb{R}$$

Your answer should include a clear statement of any conditions required for the inverse to exist.

3. Prove that $e^{\mathbf{A}}e^{\mathbf{B}} = e^{(\mathbf{A}+\mathbf{B})}$, for all $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$ such that $\mathbf{AB} = \mathbf{BA}$, for any $n \in \mathbb{N}$.

4. Find an orthogonal matrix $\mathbf{P}$ and diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^T$, where:

$$\mathbf{A} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -3 & 3 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Section 2: Advanced Calculus

Choose three of the four problems to solve.

1. Using the limit definition of the derivative, prove that $\frac{d}{dx}e^x = e^x$.
   Note: You may make use of either of the following definitions:

$$e^x \equiv \sum_{k=0}^{\infty} \frac{x^k}{k!}, \qquad\qquad e^x \equiv \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

2. Consider the spherical coordinate substitution:

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

$$r \in [0, \infty), \quad \theta \in [0, \pi], \quad \phi \in [0, 2\pi)$$

Show that the gradient of a function expressed in these coordinates is:

$$\nabla f(r, \theta, \phi) = \left[\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\right]_{r,\phi,\theta}$$

Note: An arbitrary vector in spherical coordinates is expressed in terms of unit vectors in the tangential directions of each coordinate:

$$[a, b, c]_{r,\phi,\theta} \equiv a\vec{e}_r + b\vec{e}_\theta + c\vec{e}_\phi, \quad \vec{e}_u = \left(\frac{1}{||\partial\vec{r}/\partial u||}\right)\frac{\partial\vec{r}}{\partial u}, \quad \vec{r} = [x, y, z]$$

3. A torus is a donut-shaped 3-dimensional object obtained by revolving a circle around an axis that does not intersect the circle; the minor (or cross-sectional) radius of the torus is the radius of the generating circle, and the major radius is the (minimum) distance between the center of the generating circle and the axis of revolution. Consider a torus with major radius, R, and minor radius, r.

   (a) Derive a general expression for the volume of the torus.

   (b) Derive a general expression for the surface area of the torus.

4. The plane defined by $Ax + By + Cz = 0$ intersects the ellipsoid defined by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ in an ellipse. Find the lengths of the major and minor axes of this ellipse, in terms of real variables a, b, c, A, B, C; *i.e.* find the distance between the two points furthest from (major axis) and closest to (minor axis) the origin.